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Onset of convection in a fluid layer overlying a layer of a porous medium

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A linear stability analysis is applied to a system consisting of a horizontal fluid layer overlying a layer of a porous medium saturated with the same fluid, with uniform heating from below. Surface-tension effects at a deformable upper surface are allowed for. The solution is obtained for constant-flux thermal boundary conditions.

1. Introduction

The onset of convection in a system consisting of a horizontal fluid layer overlying a porous medium saturated with that fluid is of geophysical interest. On a smaller length scale, the same problem offers the possibility of performing an experiment in order to learn something about the properties of the porous medium by observation of the convection in the fluid. In some circumstances the fluid will be a liquid, with a free surface, of sufficiently small depth for surface-tension effects to be important. The problem is also of general interest, since it involves the coupling of a sixth-order system of differential equations with one of fourth order.

Well-developed theoretical treatments of convection in a single fluid layer or porous layer are available. A convenient reference is Joseph (1976). As far as the author is aware, no theoretical work on the combined problem has been published, although he knows of two preliminary efforts which have been made by other people, who have had some difficulty with boundary conditions. Likewise there appears to be no publishable experimental data available.

In this paper we present a linear stability analysis for the problem, which is formulated in terms of quite general boundary conditions. We include the possibility of a Marangoni effect at a deformable upper surface. Since no experimental data are available, and since a large number of parameters is involved in even the simplest case, we then restrict our solution to the case of constant-flux boundary conditions. The small wavenumber expansion is then appropriate, and the solution can be obtained in simple analytic form.

2. Analysis

Basic differential equations

We take a Cartesian co-ordinate system with origin in the interface between the porous medium and the fluid layer and with the z axis vertically upwards. We suppose that the fluid layer has thickness d and the upper surface has a deflexion $h(x, y)$ from the mean. Thus the fluid occupies the region $0 \leqslant z \leqslant d+h$, where the Oberbeck–Boussinesq equations apply. We assume that non-oscillatory convection occurs. (The principle of exchange of stabilities holds for a single layer of either fluid or a porous medium, and there is no apparent reason why it should not hold here.) Hence it is sufficient to consider the steady-state equations for the fluid:

$$\operatorname{div} \mathbf{u} = 0, \quad \mathbf{u} = (u, v, w), \tag{1}$$

$$\mathbf{u} . \nabla \mathbf{u} = -\rho_0^{-1} \nabla P + \nu \nabla^2 \mathbf{u} - g[1-\alpha(T-T_0)]\, \mathbf{e}_z, \tag{2}$$

$$\mathbf{u} . \nabla T = \kappa \nabla^2 T, \tag{3}$$

where $\mathbf{u}$ = velocity, T = temperature, P = pressure, ρ_0 = standard density, μ = dynamic viscosity, $\nu = \mu/\rho_0$, α = coefficient of volume expansion, k = thermal conductivity, c = specific heat at constant pressure and $\kappa = k/\rho_0 c$, in the fluid.

Similarly, we suppose that the porous medium occupies the region $-d_m \leqslant z \leqslant 0$, where the steady-state Darcy–Oberbeck–Boussinesq equations are

$$\operatorname{div} \mathbf{u}_m = 0, \quad \mathbf{u}_m = (u_m, v_m, w_m), \tag{4}$$

$$0 = -\rho_0^{-1} \nabla P_m - (\nu/K)\, \mathbf{u}_m - g[1-\alpha(T_m - T_0)]\, \mathbf{e}_z, \tag{5}$$

$$\mathbf{u}_m . \nabla T_m = \kappa_m \nabla^2 T_m, \tag{6}$$

where $\mathbf{u}_m$ = seepage velocity, T_m = temperature, P_m = pressure, $k_m = \phi k + (1-\phi)\, k_s$, k_s = conductivity of solid matrix, ϕ = porosity, K = permeability and $\kappa_m = k_m/\rho_0 c$, in the porous medium.

Static, conduction solution

We suppose that the lower boundary ($z = -d_m$) is maintained at a uniform temperature T_L and the upper boundary ($z = d$) at a uniform temperature T_U. The steady-state solution is then

$$h = 0, \quad \mathbf{u} = 0, \tag{7), (8}$$

$$T = \tilde{T} \equiv T_0 - (T_0 - T_U)\, z/d, \tag{9}$$

$$P = \tilde{P} \equiv P_0 - \rho_0 g z - (\rho_0 \alpha g / 2d)\, (T_0 - T_U)\, z^2, \tag{10}$$

$$\mathbf{u}_m = 0, \tag{11}$$

$$T_m = \tilde{T}_m \equiv T_0 - (T_L - T_0)\, z/d_m, \tag{12}$$

$$P_m = \tilde{P}_m \equiv P_0 - \rho_0 g z - (\rho_0 \alpha g / 2d_m)\, (T_L - T_0)\, z^2, \tag{13}$$

where the interface temperature and pressure are

$$T_0 = \frac{k d_m T_U + k_m d T_L}{k d_m + k_m d}, \tag{14}$$

$$P_0 = P_a + \rho_0 g d + (\rho_0 \alpha g / 2d)\, (T_0 - T_U), \tag{15}$$

where P_a is the pressure at $z = d$.

Perturbation equations

We define $\theta = T - \tilde{T}$, $p = P - \tilde{P}$, $\theta_m = T_m - \tilde{T}_m$, $p_m = P_m - \tilde{P}_m$ and derive the linearized equations

$$\operatorname{div} \mathbf{u} = 0, \tag{16}$$

$$\rho_0^{-1}\nabla p - \nu\nabla^2\mathbf{u} - g\alpha\theta\mathbf{e}_z = 0, \tag{17}$$

$$(T_0 - T_U)d^{-1}w + \kappa\nabla^2\theta = 0, \tag{18}$$

$$\operatorname{div}\mathbf{u}_m = 0, \tag{19}$$

$$\rho_0^{-1}\nabla p_m + (\nu/K)\,\mathbf{u}_m - g\alpha\theta_m e_z = 0, \tag{20}$$

$$(T_L - T_0)d_m^{-1}w_m + \kappa_m\nabla^2\theta_m = 0. \tag{21}$$

Operating on (17) with $\mathbf{e}_z$. curl curl and using (16), we get

$$\nu\nabla^4 w + g\alpha\nabla_2^2\theta = 0, \tag{22}$$

where $\nabla_2^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2.$

Similarly (19) and (20) imply that

$$(\nu/K)\,\nabla^2 w_m - g\alpha\nabla_2^2\theta_m = 0. \tag{23}$$

Boundary conditions for perturbation variables

At the upper boundary $z = d + h$, we have

$$\mathbf{u}.\mathbf{n} = 0, \quad k\nabla\theta.\mathbf{n} + q\theta = 0, \tag{24), (25}$$

$$2\mu D_{nt} = \frac{\partial\sigma}{\partial T}\left[\nabla\left\{\theta - \left(\frac{T_0 - T_U}{d}\right)h\right\}\right].\mathbf{t}, \tag{26}$$

$$2\mu D_{nn} - p + \rho_0 gh + \frac{\rho_0 g\alpha}{2d}(T_0 - T_U)(2hd + h^2) = \sigma\nabla_2.\left(\frac{\nabla_2 h}{[1 + (\nabla_2 h)^2]^{\frac{1}{2}}}\right). \tag{27}$$

At the interface $z = 0$, we have

$$w = w_m, \tag{28}$$

$$\frac{\partial u}{\partial z} = \frac{\tilde{\alpha}}{K^{\frac{1}{2}}}\left(u + \frac{K}{\mu}\frac{\partial p_m}{\partial x}\right), \quad \frac{\partial v}{\partial z} = \frac{\tilde{\alpha}}{K^{\frac{1}{2}}}\left(v + \frac{K}{\mu}\frac{\partial p_m}{\partial y}\right), \tag{29), (30}$$

$$-p + 2\mu\,\partial w/\partial z = -p_m, \tag{31}$$

$$\theta = \theta_m, \tag{32}$$

$$k\,\partial\theta/\partial z = k_m\,\partial\theta_m/\partial z. \tag{33}$$

At the lower boundary $z = -d_m$, we have

$$w_m = 0, \tag{34}$$

$$k_m\,\partial\theta_m/\partial z - q_m\theta_m = 0. \tag{35}$$

Here $\mathbf{t}$ and $\mathbf{n}$ denote tangential and normal unit vectors at the surface, q and q_m are heat-transfer coefficients, σ is the surface tension, $\{D_{ij}\}$ is the rate-of-strain tensor in the fluid and ∇_2 is the horizontal Laplacian.

Equations (24) and (34) express the condition that there is no mass flux across the boundaries, (25) and (35) are radiation-type conditions, (26) and (27) express the continuity of tangential and normal stress at the upper boundary, (28), (31), (32) and (33) express the continuity of normal velocity, normal stress, temperature and heat flux at the interface, and (29) and (30) are the conditions of Beavers & Joseph (1967) relating the shear in the fluid to the slip velocity at the interface; $\tilde{\alpha}$ is their constant.

In writing (26) we have assumed that the basic, conduction thermal distribution applies right up to the deformed boundary. This assumption is consistent with the constant-flux condition [(25) with $q = 0$] but when the general thermal condition (25) with $q \neq 0$ applies it may be preferable to make an alternative assumption.

When linearized, (24)–(27) become, at $z = d$,

$$w = 0, \tag{36}$$

$$k\,\partial\theta/\partial z + q\theta = 0, \tag{37}$$

$$\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) = \frac{\partial \sigma}{\partial T}\left[\frac{\partial \theta}{\partial x}-\left(\frac{T_0-T_U}{d}\right)\frac{\partial h}{\partial x}\right], \tag{38}$$

$$\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) = \frac{\partial \sigma}{\partial T}\left[\frac{\partial \theta}{\partial y}-\left(\frac{T_0-T_U}{d}\right)\frac{\partial h}{\partial y}\right], \tag{39}$$

$$2\mu\,\partial w/\partial z - p + \rho_0 g h = \sigma \nabla_2^2 h. \tag{40}$$

(We have assumed that $\alpha(T_0 - T_U) \ll 1$ in writing (40). This is in keeping with the Boussinesq approximation.) We can use the continuity equation (16) to write (38) and (39) as the single equation

$$\mu\frac{\partial^2 w}{\partial z^2} = \frac{\partial \sigma}{\partial T}\left[\nabla_2^2\theta-\left(\frac{T_0-T_U}{d}\right)\nabla_2^2 h\right]. \tag{41}$$

Similarly, using (16) and (19) we replace (27) and (28) by

$$\frac{\partial^2 w}{\partial z^2} = \frac{\tilde{a}}{K^{\frac{1}{2}}}\left(\frac{\partial w}{\partial z}-\frac{\partial w_m}{\partial z}\right). \tag{42}$$

Finally, we eliminate p and p_m from (31) by using (17), (20), (16) and (19) to obtain

$$\left(\frac{\partial^2}{\partial z^2}+3\nabla_2^2\right)\frac{\partial w}{\partial z} = -\frac{1}{K}\frac{\partial w_m}{\partial z}. \tag{43}$$

Status of the equations

The Oberbeck–Boussinesq equations (1)–(3) are very well known, and equations (4)–(6) are also well established. The boundary conditions (24)–(27) are essentially those used by Smith (1966). Predictions based on the Pearson boundary condition (26) have been compared with experimental results obtained by several workers including Palmer & Berg (1971), and the agreement is generally satisfactory. With the exception of the Beavers–Joseph conditions (38) and (39), the remaining boundary conditions are routine and require no further comment.

The Beavers–Joseph condition was first presented as an empirical result, but a certain amount of theoretical foundation for it has since been provided, and several subsequent experimental investigations have provided further support (see Neale & Nader 1974; Beavers, Sparrow & Masha 1974, and the references contained therein). Although other models for the porous-medium interface, e.g. that of Ene & Sanchez-Palencia (1975), lead to an alternative boundary condition, the Beavers–Joseph condition appears to be the best established at the present time. The results obtained in the present paper lay the basis for further experimental testing of the usefulness of this condition, since the other elements in our model are already reasonably well established.

Non-dimensional formulation

We choose separate scales for the fluid layer and porous medium,† so that our equations exhibit what symmetry there is in the problem; we note that there is no natural horizontal length scale here.

For the fluid layer we write

$$\mathbf{u} = \frac{\kappa}{d}\mathbf{u}', \quad \theta = (T_0 - T_U)\theta', \quad (x, y, z, h) = d(x', y', z', h'), \quad p = \frac{\mu\kappa}{d^2}p'.$$

For the porous medium we write

$$\mathbf{u}_m = \frac{\kappa_m}{d_m}\mathbf{u}_m', \quad \theta_m = (T_L - T_0)\theta', \quad (x, y, z) = d_m(x_m', y_m', z_m' - 1), \quad p_m = \frac{\mu\kappa_m}{d_m^2}p_m'.$$

We substitute in the differential equations (22), (18), (23) and (21) and drop primes, obtaining

$$\left.\begin{aligned} \nabla^4 w + R\nabla_2^2\theta &= 0 \\ w + \nabla^2\theta &= 0 \end{aligned}\right\} \quad \text{in} \quad 0 \leqslant z \leqslant 1, \qquad \begin{aligned}(44)\\(45)\end{aligned}$$

$$\left.\begin{aligned} \nabla_m^2 w_m - R_m\nabla_{m2}^2\theta_m &= 0 \\ w_m + \nabla_m^2\theta_m &= 0 \end{aligned}\right\} \quad \text{in} \quad 0 \leqslant z_m \leqslant 1, \qquad \begin{aligned}(46)\\(47)\end{aligned}$$

where

$$R = \frac{g\alpha(T_0 - T_U)\,d^3}{\kappa\nu}, \quad R_m = \frac{g\alpha(T_L - T_0)\,Kd_m}{\kappa_m\nu}$$

are the Rayleigh and Rayleigh–Darcy numbers for the two layers, and ∇_m denotes the gradient with respect to co-ordinates with subscript m.

Similarly our boundary conditions may be written as follows. At $z = 1$,

$$w = 0, \tag{48}$$

$$\partial\theta/\partial z + B\theta = 0, \tag{49}$$

$$\partial^2 w/\partial z^2 - M(\nabla_2^2\theta - \nabla_2^2 h) = 0, \tag{50}$$

$$\frac{\partial^3 w}{\partial z^3} + 3\nabla_2^2\frac{\partial w}{\partial z} + G\nabla_2^2 h - S\nabla_2^4 h = 0. \tag{51}$$

At $z = 0$, or $z_m = 1$,

$$\hat{T}w = w_m, \tag{52}$$

$$\hat{T}\hat{d}\left(\frac{\partial w}{\partial z} - \frac{\beta\hat{d}}{\tilde{\alpha}}\frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial w_m}{\partial z_m}, \tag{53}$$

$$\hat{T}\beta^2\hat{d}^3\left(\frac{\partial^3 w}{\partial z^3} + 3\nabla_2^2\frac{\partial w}{\partial z}\right) = -\frac{\partial w_m}{\partial z}, \tag{54}$$

$$\theta = \hat{T}\theta_m, \quad \partial\theta/\partial z = \partial\theta_m/\partial z_m. \qquad (55), (56)$$

† This type of scaling is particularly useful for the related problems involving two fluid layers, or two porous layers. The author has performed, but not published, the analogous analysis for these problems.

At $z_m = 0$,
$$w_m = 0, \tag{57}$$
$$\partial\theta_m/\partial z_m - B_m\theta_m = 0. \tag{58}$$

Here $B = qd/k, \quad B_m = q_m d_m/k_m$ are Biot numbers,

$$M = -\frac{\partial\sigma}{\partial T}\frac{(T_0 - T_U)\,d}{\mu\kappa}$$ is the Marangoni number,

$G = gd^3/\nu\kappa$ is a gravitational number,

$S = \sigma d/\mu\kappa$ is a surface-tension number,

while
$$\hat{T} = \frac{T_L - T_0}{T_0 - T_U}, \quad \hat{d} = \frac{d_m}{d}, \quad \beta = \frac{\sqrt{K}}{d_m}.$$

We have used the fact that $\kappa_m/\kappa = k_m/k = \hat{d}/\hat{T}$, which follows since the steady-state heat flux is continuous across the interface.

Normal-mode expansion

We write
$$\begin{pmatrix} w \\ \theta \\ h \end{pmatrix} = \begin{pmatrix} W(z) \\ \Theta\,(z) \\ H\,(z) \end{pmatrix} f(x,y), \quad \begin{bmatrix} w_m \\ \theta_m \end{bmatrix} = \begin{bmatrix} W_m(z) \\ \Theta_m(z) \end{bmatrix} f_m(x_m, y_m), \tag{59}$$
where
$$\nabla_2^2 f + a^2 f = 0, \quad \nabla_{m2}^2 f_m + a_m^2 f_m = 0. \tag{60}$$

The separation constants a and a_m are non-dimensional horizontal wavenumbers. Since the dimensional horizontal wavenumber must be the same for the fluid and porous medium if matching of solutions in the two layers is to be possible, we must have $a/d = a_m/d_m$, and hence $a_m = \hat{d}a$.

We write D for the derivative d/dz and D_m for d/dz_m. We then have
$$(D^2 - a^2)^2\,W - Ra^2\Theta = 0 \quad \text{in} \quad 0 \leqslant z \leqslant 1, \tag{61}$$
$$W + (D^2 - a^2)\,\Theta = 0 \quad \text{in} \quad 0 \leqslant z \leqslant 1, \tag{62}$$
$$(D_m^2 - a_m^2)\,W_m + R_m a_m^2\,\Theta_m = 0 \quad \text{in} \quad 0 \leqslant z_m \leqslant 1, \tag{63}$$
$$W_m + (D_m^2 - a_m^2)\,\Theta_m = 0 \quad \text{in} \quad 0 \leqslant z_m \leqslant 1, \tag{64}$$

subject to
$$W(1) = 0, \quad D\Theta(1) + B\Theta(1) = 0, \tag{65), (66}$$
$$D^2W(1) + Ma^2[\Theta(1) - H] = 0, \tag{67}$$
$$D^3W(1) - 3a^2DW(1) - [Ga^2 + Sa^4]\,H = 0, \tag{68}$$
$$\hat{T}\,W(0) = W_m(1), \tag{69}$$
$$\hat{T}\hat{d}[DW(0) - (\beta\hat{d}/\tilde{\alpha})\,D^2W(0)] = D_m W_m(1), \tag{70}$$
$$T\beta^2\hat{d}^3[D^3W(0) - 3a^2DW(0)] = -D_m W_m(1), \tag{71}$$
$$\Theta(0) = \hat{T}\Theta_m(1), \quad D\Theta(0) = D_m\,\Theta_m(1), \tag{72), (73}$$
$$W_m(0) = 0, \quad D_m\,\Theta_m(0) - B_m\,\Theta_m(0) = 0. \tag{74), (75}$$

We thus have a tenth-order system of differential equations and, when H is eliminated, ten boundary conditions, forming a standard eigenvalue problem, whose solution is routine but tedious. Noting that $R_m = \hat{T}^2\beta^2\hat{d}^2R$ and $a_m = \hat{d}a$, we see that we can regard R as the eigenvalue and a as a parameter. For the stability criterion we must minimize R as a function of a. In view of the large number of parameters involved, and the absence of experimental data, it seems sensible to postpone any large-scale computation, and instead look at the case of constant-heat-flux boundary conditions. Then the minimization process can be effectively avoided, since we expect the critical wavenumber (which gives the minimum R) to be zero as it is in the case of a single layer of fluid or porous medium. As a bonus we get a stability criterion in simple algebraic form, so that the effect of varying parameters can be quickly investigated.

3. Solution for constant-heat-flux boundary conditions

We take $B = B_m = 0$, and expand in terms of a small wavenumber a. We recall that $a_m = \hat{d}a$, and write

$$\begin{bmatrix} W \\ \Theta \\ H \\ R \end{bmatrix} = \sum_{j=0}^{\infty} a^{2j} \begin{bmatrix} W_j \\ \Theta_j \\ H_j \\ R_j \end{bmatrix}, \quad \begin{bmatrix} W_m \\ \Theta_m \\ R_m \end{bmatrix} = \sum_{j=0}^{\infty} (\hat{d}a)^{2j} \begin{bmatrix} W_{mj} \\ \Theta_{mj} \\ R_{mj} \end{bmatrix}. \tag{76}$$

The equations at *order zero* are

$$\left.\begin{aligned} D^4W_0 &= 0 \\ W_0 + D^2\Theta_0 &= 0 \end{aligned}\right\} \text{ in } 0 \leqslant z \leqslant 1, \qquad \left.\begin{aligned} D_m^2 W_{m0} &= 0 \\ W_{m0} + D_m^2\Theta_{m0} &= 0 \end{aligned}\right\} \text{ in } 0 \leqslant z_m \leqslant 1, \tag{77}$$

$$\left.\begin{gathered} W_0(1) = 0, \quad D\Theta_0(1) = 0, \quad D^2W_0(1) = 0, \quad D^3W_0(1) = 0, \\ \hat{T}W_0(0) = W_{m0}(1), \quad \Theta_0(0) = \hat{T}\Theta_{m0}(1), \quad D\Theta_0(0) = D_m\Theta_{m0}(1), \\ \hat{T}\hat{d}[DW_0(0) - (\beta\hat{d}/\tilde{\alpha})\, D^2W_0(0)] = D_mW_m(1) = -\hat{T}\beta^2\hat{d}^3W_0(0), \\ W_{m0}(0) = 0, \quad D_m\Theta_{m0}(0) = 0. \end{gathered}\right\} \tag{78}$$

To within an arbitrary factor, the solution is

$$W_0(z) = 0, \quad \Theta_0(z) = \hat{T}, \quad W_{m0}(z) = 0, \quad \Theta_{m0}(z) = 1.$$

Using these results, the equations of *order* a^2 become

$$\left.\begin{aligned} D^4W_1 &= R_0\hat{T} \\ W_1 + D^2\Theta_1 &= \hat{T} \end{aligned}\right\} \text{ in } 0 \leqslant z \leqslant 1, \qquad \left.\begin{aligned} D_m^2 W_{m1} &= -R_m \\ W_{m1} + D_m^2\Theta_{m1} &= 1 \end{aligned}\right\} \text{ in } 0 \leqslant z_m \leqslant 1, \tag{79}$$

$$\left.\begin{gathered} W_1(1) = 0, \quad D\Theta_1(1) = 0, \quad D^2W_1(1) + M_0(T - H_0) = 0, \quad D^3W_1(1) - GH_0 = 0, \\ \hat{T}W_1(0) = \hat{d}^2W_{m1}(1), \quad \Theta_1(0) = \hat{T}\hat{d}^2\Theta_{m1}, \quad D\Theta_1(0) = \hat{d}^2D_m\Theta_{m1}(1), \\ \hat{T}\hat{d}(DW_1(0) - (\beta\hat{d}/\tilde{\alpha})\, D^2W_1(0)) = \hat{d}^2D_mW_{m1}(1) = -\hat{T}\beta^2\hat{d}^3D^3W_1(0), \\ W_{m1}(0) = 0, \quad D_m\Theta_{m1}(0) = 0. \end{gathered}\right\} \tag{80}$$

The differential equations (79) have a general solution of the form

$$\left.\begin{gathered} W_1 = \tfrac{1}{24}R_0\hat{T}(z^4 + c_3z^3 + c_2z^2 + c_1z + c_0), \\ \Theta_1 \doteq \hat{T}(\tfrac{1}{2}z^2 + d_1z + d_0) - \tfrac{1}{24}R_0\hat{T}(\tfrac{1}{30}z^6 + \tfrac{1}{20}c_3z^5 + \tfrac{1}{12}c_2z^4 + \tfrac{1}{6}c_1z^3 + \tfrac{1}{2}c_0z^2), \\ W_{m1} = -R_{m0}(\tfrac{1}{2}z_m^2 + e_1z_m + e_0), \\ \Theta_{m1} = \tfrac{1}{2}z_m^2 + f_1z_m + f_0 + R_m(\tfrac{1}{24}z_m^4 + \tfrac{1}{6}e_1z_m^3 + \tfrac{1}{2}e_0z_m^2). \end{gathered}\right\} \tag{81}$$

We note that only one of the boundary conditions (80) involves the constants d_0 and f_0. Hence the temperature perturbation is left indeterminate to within a constant. The three boundary conditions involving $D\Theta_1$ and $D_m\Theta_{m1}$ and the differential equations involving $D^2\Theta_1$ and $D^2\Theta_{m1}$ yield

$$\int_0^1 W_1\,dz + \hat{d}^2 \int_0^1 W_{m1}\,dz_m = \hat{T} + \hat{d}^2. \tag{82}$$

The remaining seven boundary conditions determine H_0 and the constants c_0, c_1, c_2, c_3, e_0 and e_1. Thus W_1 and W_{m1} are determined and their expressions can be substituted into (82) to get an equation determining the eigenvalue. We thus obtain the result

$$\begin{aligned} &R\{3-12\mu+(24-84\mu)\,\lambda+\tau[84-240\mu+(384-960\mu)\,\hat{d}+(300-720\mu)\,\hat{T}\hat{d} \\ &\quad +(720-1440\mu)\,\lambda(\hat{d}+\hat{T}\hat{d})]\}+R_m\hat{d}^2\hat{T}^{-2}\{300-480\mu+(320-480\mu) \\ &\quad \times\hat{T}+(720+960\hat{T})\,(1-\mu)\,\lambda+\tau[720+960\hat{T}+240\hat{T}\hat{d}]\} \\ &\quad +M\{20+120\lambda+\tau[240+960\hat{d}+720\hat{T}\hat{d}+1440(1+\hat{T})\,\lambda]\} \\ &= (1+\hat{d}^2\hat{T}^{-1})\,[960-1440\mu+2880(1-\mu)\,\lambda+2880\tau(1+\hat{d})], \end{aligned} \tag{83}$$

where here $\qquad \mu = M/G, \quad \lambda = \beta\hat{d}/\tilde{\alpha}, \quad \tau = \beta^2\hat{d}^2.$

We can check our formula against known results for some special cases (Nield 1967, 1968).

(i) Letting $\mu \to 0$, $\hat{d} \to 0$ and $\hat{k} \to 0$ with $\hat{T} = \hat{d}/\hat{k}$ finite, we get

$$\tfrac{1}{320}R + \tfrac{1}{48}M = 1,$$

the known result for a viscous fluid between one rigid and one free boundary.

(ii) Letting $\mu \to 0$, $\hat{d} \to 0$ and $\lambda \to \infty$, we get

$$\tfrac{1}{120}R + \tfrac{1}{24}M = 1,$$

the result for a viscous fluid between two free boundaries.

(iii) Letting $\mu \to 0$, $\hat{d} \to \infty$, $\hat{T} \to \infty$, $\lambda \to \infty$ and $\tau \to 0$, we get

$$R_m = 3,$$

the known result for a porous medium between one impermeable boundary and one boundary at constant pressure.

(iv) Letting $\mu \to 0$, $\hat{d} \to \infty$, $\hat{T} \to 0$ and $\tau \to \infty$, we get

$$R_m = 12,$$

the result for a porous medium between two impermeable boundaries.

For most practical situations the value of

$$\mu = -\frac{\partial\sigma}{\partial T}\frac{(T_0 - T_U)}{\rho g d^2}$$

will be negligibly small compared with unity. Also we know that $\beta = K^{\frac{1}{2}}/d_m$ will normally be a small quantity, while $\tilde{\alpha}$ is of order unity. Hence, unless $\hat{d} = d_m/d$ is large, the quantities λ and τ will be small, and so will $R_m/R = \hat{T}^2\beta^2\hat{d}^2$, so that, to first order in β, equation (83) reduces to

$$\left(\frac{1+8\lambda}{1+3\lambda}\right)\frac{R}{320} + \left(\frac{1+6\lambda}{1+3\lambda}\right)\frac{M}{48} = 1 + \frac{\hat{d}^2}{\hat{T}}. \tag{84}$$

As we should expect, the effect of increasing λ is to reduce the critical value of R (or M) at which convection appears.

It may not always be practical to measure $T_0 - T_U$ and $T_L - T_0$ separately; usually the total temperature difference $T_L - T_U = \Delta T$ will be measured. Equation (84) can be rearranged as

$$\Delta T = \frac{960(1+3\lambda)\rho\nu(\kappa d + \kappa_m d_m)(\kappa_m d + \kappa d_m)}{3(1+8\lambda)\rho g \alpha d^5 + 20(1+6\lambda)(-\partial\sigma/\partial T)d^3}. \tag{85}$$

Similarly, the general relationship (83) gives ΔT in terms of d, d_m and the properties of the fluid and porous medium.

To complete the calculation of the eigenfunctions W, W_m, Θ and Θ_m is simple but tedious, and we do not present the result. We already know what to expect. Since k_m/k is likely to be of order unity in practical situations, the temperature perturbation (Θ, Θ_m) will be a continuous function with a single peak occurring more or less midway between the boundaries, and hence in the layer of greater depth. In contrast, β will usually be small, and (54) implies that then the continuous function (W, W_m) will have its peak near the interface.

The deflexion of the upper free surface is a quantity of interest. We find that

$$H_0 = \frac{3(1+4\lambda+8\tau(1+\hat{d})]\hat{T}R + 12\hat{T}^{-1}\hat{d}^2 R_m - 12(1+2\lambda)\hat{T}M}{8G\{1+3\lambda+3\tau(1+\hat{d}) - 12\mu(1+2\lambda)\}}. \tag{86}$$

Thus if $\mu = M/G$ is small, H_0 is positive for buoyancy-induced convection ($R, R_m \neq 0$, $M = 0$) and negative for surface-tension-induced convection ($R = R_m = 0$, $M \neq 0$), while the signs are reversed if μ is sufficiently large, in accordance with the results of Scriven & Sterling (1964) and Smith (1966).

4. Conclusion

We have solved our problem for the case of constant-flux boundary conditions. Results obtained for this special case should be qualitatively useful for estimating the stability criterion for more general thermal boundary conditions, when the critical wavenumber will no longer be zero, and when the surface-tension parameter S will play a part. The effects of S can be estimated from our present results by noting that it enters only in the combination $G + a^2 S$ [through condition (68)] and that a_c will usually be of order unity. Hence, to a first approximation, the effects of S and G are additive.

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